



TITLE:

Compactifying Spec Z

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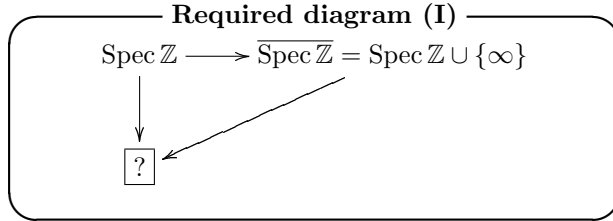
RIGHT:

COMPACTIFYING $\text{Spec } \mathbb{Z}$

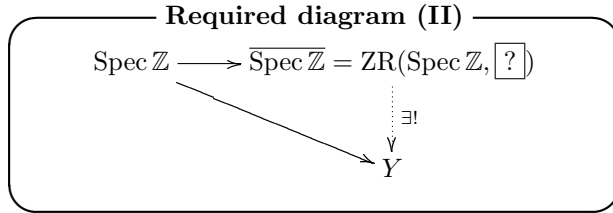
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1 Motivation

We want a category in which we can compactify $\text{Spec } \mathbb{Z}$:



We want $\overline{\text{Spec } \mathbb{Z}}$ to be the *Zariski-Riemann space* $\text{ZR}(\text{Spec } \mathbb{Z}, \boxed{?})$ of $\text{Spec } \mathbb{Z}$, namely to have the universal property as follows: $\text{Spec } \mathbb{Z} \rightarrow \boxed{?}$ is proper, and if $\text{Spec } \mathbb{Z} \rightarrow Y$ is a morphism over $\boxed{?}$ and $Y \rightarrow \boxed{?}$ is proper, then there should exist a unique morphism $\overline{\text{Spec } \mathbb{Z}} \rightarrow Y$:



However, these diagrams cannot be obtained in the category **(Sch)** of schemes:

- (1) $\text{Spec } \mathbb{Z}$ is the initial object in **(Sch)**, hence we don't have any object $\boxed{?}$ under $\text{Spec } \mathbb{Z}$.
- (2) The infinite place ∞ is not a prime ideal.

Several attempts have been made in the past (Haran, Durov, ...), but have not reached the characterization of $\overline{\text{Spec } \mathbb{Z}}$ by the universal property.

Therefore, we must consider an algebraic type which generalizes that of commutative rings; this is what we call 'convexoid rings', as is defined below.

2 Definitions

An algebraic type τ is *commutative*, if any m -ary operator φ and n -ary operator ψ commutes:

$$\begin{aligned} \varphi(\psi(x_{11}, \dots, x_{1n}), \dots, \psi(x_{m1}, \dots, x_{mn})) \\ = \psi(\varphi(x_{11}, \dots, x_{m1}), \dots, \varphi(x_{1n}, \dots, x_{mn})) \end{aligned}$$

This enables us to give a τ -algebra structure on Hom sets, and hence to define tensor products \otimes .

A *convexoid* is a quadruple $(M, \boxplus, -, 0)$ which satisfies:

- (1) M is a set, $0 \in M$, \boxplus (resp. $-$) is a binary (resp. unary) operator on M ,

- (2) The algebraic structure on M is commutative in the above sense, and
- (3) $a \boxplus b = b \boxplus a$, $(-a) \boxplus a = 0$.

A *convexoid ring* is a commutative monoid object with respect to \otimes in the category of convexoids.

Example 2.0.1. $\mathbf{DQ} = \{x \in \mathbb{Q} \mid |x| \leq 1\}$ is a convexoid ring, by setting $a \boxplus b = (a + b)/2$. Note that \boxplus is **not associative**.

We can define 'convexoid schemes', just as in the way of usual schemes; however, this is **NOT sufficient** for our purpose.

If R is a convexoid ring, then $\gamma_R = 1 \boxplus 0$ is the *fundamental constant*.

A ring is a localization of a convexoid

$$\begin{aligned} R[\gamma_R^{-1}] \text{ becomes a ring, by setting} \\ a + b = \gamma_R^{-1}(a \boxplus b). \end{aligned}$$

A morphism $A \rightarrow B$ of convexoid rings is an *equivalence*, if $\gamma_A B = \gamma_B B$ and $A[\gamma_A^{-1}] \rightarrow B[\gamma_B^{-1}]$ is a ring isomorphism.

A *convexoid scheme* is a (multiplicative) **monoid**-valued space X which is locally isomorphic to the spectrum of some convexoid ring, and transition maps are **equivalences**.

3 Results

Let R_0 be the initial object in the category of convexoid rings. This is a submonoid of the polynomial ring $\mathbb{Z}[\gamma]$, and hence equipped with a canonical grading. We can define a convexoid scheme $\text{Proj } R_0$.

Valuation convexoid rings can be defined just like valuation rings. A *proper morphism* of convexoid schemes is defined by the valuative criterion.

Main theorem(T-) [1]

$$\overline{\text{Spec } \mathbb{Z}} = \text{ZR}(\text{Spec } \mathbb{Z}, \text{Proj } R_0).$$

This is realized in the **pro-category** of convexoid schemes.

Remark 3.0.2. • The infinity place canonically appears, **without using the terminology of norms**. This can be shown by proving a generalization of Ostrowski's theorem.

- The above theorem can be generalized to any ring of algebraic integers \mathcal{O}_K .
- A locally free sheaf on $\overline{\text{Spec } \mathcal{O}_K}$ is, **by definition** a projective \mathcal{O}_K -module equipped with a norm satisfying some finiteness property.

References

- [1] Takagi, S.: *Compactifying $\text{Spec } \mathbb{Z}$* , arXiv: math.AG/1203.4914